# GEOMETRICAL ACOUSTICS APPROXIMATION FOR SURFACE WAVES

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1. The behavior of waves on the surface of a viscous conducting fluid in external fields is of constant interest (see [1-4] for example). However, as a rule, the study is limited to uniform fields. In real problems the fields are always nonuniform.

Here a method is presented for studying the interaction of capillary waves on the surface of a viscous conducting fluid with spatially nonuniform surface stresses. Obviously, the solution to this problem is difficult in the general case. However, simplifications are possible when the field changes slowly on scales of the order of a capillary wavelength, that is, within the limits which by tradition can be called geometrical acoustics or optics. Initially the method will be summarized, and the geometrical acoustics equations will be obtained. Then the focusing properties of a large-scale nonuniformity will be studied.

The condition [4, 5] for applying geometrical acoustics and optics is that the wavelength must be small compared to the characteristic dimensions of the problem. In the case of capillary waves, this is in comparison to the scales of nonuniformity of the surface stresses or the electrical field.

2. Before we examine the geometrical acoustics equations of capillary waves in an electric field, we will study the simpler, but still useful method of the equations in the absence of a field. In this formulation of the problem, the surface stress is spatially nonuniform. Gradients in the surface stress are known [5] to lead to near-surface flows — the capillary waves propagate namely in the background of these flows. In order to examine the possibility of separating the two motions — the steady state flow and the capillary waves — we use the linearized Navier–Stokes equation and the continuity equation

$$\rho\partial\mathbf{v}/\partial t = -\nabla p + \eta\Delta\mathbf{v},\ \operatorname{div}\mathbf{v} = 0, \tag{2.1}$$

where $\mathbf{v}$ is the vector velocity, p is the pressure, $\eta$ is the viscosity, and $\rho$ is the density; the boundary conditions on the horizontal surface of the fluid (the xy plane):

$$-p + 2\eta\partial v_z/\partial z - \gamma\partial^2\xi/\partial x^2 = 0; \tag{2.2}$$

$$\eta(\partial v_x/\partial z + \partial v_z/\partial x) = \partial\gamma/\partial x \tag{2.3}$$

for z = 0. Here $\xi$ is the displacement of a point on the fluid surface from the equilibrium position; therefore $\partial\xi/\partial t = v_z|_{z=0}$.

The last term in (2.2) is the Laplace pressure ($\gamma$ is the surface tension coefficient). The boundary condition (2.3) considers the tangential component of the forces due to the surface nonuniformity.

We note that Eqs. (2.1) can be linearized under the usual assumption that the oscillation amplitudes are small compared to the characteristic wavelengths: $\xi \ll \lambda$.

We will attempt to represent the velocity as the sum $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_+$, where $\mathbf{v}_0$ is the steady-state flow velocity and $\mathbf{v}_+$ is the fluid velocity related to the wave motion.

The equations for the two motions can be separated if the phase velocity of the wave is much larger than the steady-state flow velocity. In the low-viscosity approximation [5], we have $v_+ = \omega/k = \sqrt{\gamma k/\rho}$, where $\omega$ and k are the frequency and wave number of the wave. From this it is obvious that the inequality $v_+ \gg v_0$ is fulfilled only for short waves, when $k \gg \rho v_0^2/\gamma$.

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It should be noted that we did not consider the change in mass (per unit area) related to the presence of absorbed molecules. Moreover, the boundary conditions for the equations of motion do not consider the excess surface momentum and conditions related to the difference in material density near the surface from the average volume density. This means that the thickness of fluid induced into motion can be larger than the surface transition layer, which is equivalent to the limitation $k\delta \ll 1$, where $\delta$ is the thickness of the surface layer.

We restrict ourselves to the limit of a nonviscous fluid. (We recall [6] that in geometrical optics only transparent media are examined.) Then a very simple approach can be used to describe the fluid dynamics with the use of a velocity potential $\varphi$ (where $v = \nabla\varphi$). Then (in the limit of an incompressible fluid), the condition

$$\Delta\varphi = 0 \tag{2.4}$$

is fulfilled, and moreover on the fluid surface

$$\rho\partial\varphi/\partial t|_{z=0} - \gamma\Delta_n\xi = 0, \tag{2.5}$$

where $\Delta_n = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and $(x, y)$ is the system of rectangular coordinates on the surface plane of the fluid. The relationship between the displacement from equilibrium and the potential $\varphi$ is

$$\partial\xi/\partial t = \partial\varphi/\partial z|_{z=0}. \tag{2.6}$$

In the case $\gamma$ = const, substituting the solution $\varphi = \varphi_0 \exp(-i\omega t + ikx + iKz)$ yields the dispersion equation for the capillary waves: $\omega^2 = \gamma k/\rho$.

Having assumed, however, that $\gamma(x, y)$ varies from point to point, we will examine the propagation of a wave which satisfies the inequality $kL \gg 1$, which coincides to the range of application of geometrical optics ($L$ is the characteristic scale for the change in $\gamma$). The frequency is a constant for a monochromatic wave under stationary conditions; therefore, the dependence of $\varphi$ and $\xi$ on time and the coordinates is expressed in the form

$$\begin{aligned} \xi &= \xi_0 \exp(-i\omega t + i\psi_\xi(x, y)), \\ \varphi &= \varphi_0 \exp(-i\omega t + i\psi_\varphi(x, y) + k(x, y)z). \end{aligned} \tag{2.7}$$

Here the limitation $k > 0$ must be imposed, because the wave should attenuate into the depth of the fluid ($z < 0$). In analogy to the theory of geometrical optics of electromagnetic waves, the quantities $\psi_\varphi$ and $\psi_\xi$ are called the characteristics [eikonals]. As we will see later, in the geometrical optics approximation, $\psi_\varphi$ and $\psi_\xi$ coincide with each other (with logarithmic accuracy), and $\varphi_0$ and $\xi_0$ are constant coefficients.

Before we substitute (2.7) into (2.4), we calculate $\Delta_n\varphi$:

$$\Delta_n\varphi = [i^2(\nabla_n\psi_\varphi)^2 + i\Delta_n\psi_\varphi + 2i\nabla_n\psi_\varphi\nabla_n k + z^2(\nabla_n k)^2 + z\Delta_n k]\varphi,$$

where $\Delta_n$ is a two-dimensional gradient. The characteristic $\psi_\varphi$ changes by $2\pi$ over a wavelength: $\partial\psi_\varphi/\partial x \sim 2\pi/\lambda$, which means that from $\psi_\varphi \sim 2\pi x/\lambda$ for capillary waves ($\lambda \to 0$) it follows that $\psi_\varphi$ is a large quantity. Therefore, only the first term is retained in $\Delta_n\varphi$. Thus, Eq. (2.4) takes the form

$$-(\nabla_n\psi_\varphi)^2 + k^2 = 0. \tag{2.8}$$

After substitution of $\varphi$ and $\xi$, Eq. (2.6) is represented as

$$-i\omega\xi_0 \exp(i\psi_\xi(x, y)) = k(x, y)\varphi_0 \exp(i\psi_\varphi(x, y)).$$

By using $k_0$ [the value of $k(x, y)$ at infinity] and by introducing the equality $k_0\varphi_0 = -i\omega\xi_0$, we write the relationship between $\psi_\varphi$ and $\psi_\xi$: $n(x, y)\cdot\exp(i\psi_\varphi) = \exp(i\psi_\xi)$,

where, in analogy to optics, the index of refraction is

$$n = k(x, y)/k_0. \tag{2.9}$$

By considering that $\psi_\varphi$ and $\psi_\xi$ have large values, it can be concluded that they are equal (to an accuracy of $\ln n$). Then Eq. (2.5) is represented in the form

$$-\rho\omega^2 + \gamma(x, y)k(x, y)(\nabla_n\psi_\varphi)^2 = 0.$$

By combining Eqs. (2.8) and (2.9) we obtain

$$(\nabla_n\psi_\varphi)^2 = k_0^2 n^2, \text{ where } n = (\gamma_0/\gamma(x, y))^{1/3}, \tag{2.10}$$

while

$$k_0 = (\rho\omega^2/\gamma_0)^{1/3}, \tag{2.11}$$

and $\gamma_0$ is the surface tension coefficient at infinity. Equation (2.10) coincides in form to the characteristic equation from [6], which determines the propagation of rays in a medium with an index of refraction n. By using [6], we can write the equation for finding the form of the rays:

$$d\mathbf{l}/dl = [\nabla_n n - \mathbf{l}(\mathbf{l}\nabla_n n)]/n, \tag{2.12}$$

where $\mathbf{l}$ is the unit vector tangent to the ray and the derivative is calculated along the ray trajectory.

It must be noted that in order to separate the description of the steady-state and wave motions, another inequality must be fulfilled besides the condition $v_+ \gg v_0$. In studying the motion of the capillary waves through a nonuniform surface stress, we did not consider the effect of the steady-state flow. This is possible if the change in the velocity of the capillary wave, which depends on the surface stress, is much larger than the steady-state flow velocity, that is, $\Delta v_+ \gg v_0$.

From $v_+ = \sqrt{\gamma k/\rho}$ we have

$$\Delta v_+ = (\partial\gamma/\partial x)L/2\sqrt{k/\rho\gamma}.$$

The velocity $v_0$ can be estimated from (2.3) by assuming that $v_x \sim v_0$ changes markedly over a distance on the order of $\delta$, where $\delta$ is the scale of the region which is enrapped by the steady-state motion (for example the layer thickness of the fluid; for a bottomless fluid $\delta \sim L$). Therefore $\partial v_x/\partial z \sim v_0/\delta$, from which it then follows that $v_0 \sim (\delta/\eta)\partial\gamma/\partial x$. Finally, the desired inequality takes the form

$$k \gg 4\gamma\rho(\delta/\eta L)^2.$$

By using Eqs. (2.10) and (2.12), we solve the problem of geometrical acoustics for capillary waves in the absence of a field.

3. We now examine the consequences of including an electric field. As before, in this case it is necessary to separate the steady-state regime and the wave propagation. The Navier–Stokes equation, the continuity equations, and the boundary condition (2.2) remain as before. In analogy to the previously presented change in the boundary condition and the inclusion of a nonuniform surface tension, condition (2.3) takes the form

$$\eta(\partial v_x/\partial z + \partial v_z/\partial x) = \sigma E_x$$

for the nonuniform field, where $\sigma$ is the surface charge density. In the steady-state mode, the right side is zero, because the conducting surface is an equipotential surface. However, for oscillating motion, we have $E_x = E_0\partial\xi/\partial x$. From the last boundary condition it can be seen that in order to neglect the additional terms compared to the others for this condition, the inequality

$$\eta k v_x \gg k\xi 4\pi\sigma^2, \text{ otherwise } \omega \gg 4\pi\sigma^2/\eta$$

must be satisfied. By considering the expression for $\omega$, for example in the limit of low viscosity, we obtain that the last inequality is fulfilled for short wavelengths:

$$k\sqrt{(\gamma k - 4\pi\sigma^2)/\rho} \gg 4\pi\sigma^2/\eta.$$

In this case Eq. (2.5) is written in the form [6]

$$\rho\partial\varphi/\partial t|_{z=0} - \gamma\Delta_n\xi - 4\pi\sigma^2 k(x, y)\xi = 0,$$

where now $\gamma$ is a constant quantity.

The expressions (2.7), Eq. (2.8), the relationship between $\varphi_0$ and $\xi_0$, and the equation for n are used for $\varphi$ and $\xi$. We also retain the conclusion on the equality of the characteristics $\psi_\varphi$ and $\psi_\xi$. However, now the condition for the surface changes form:

$$-\rho\omega^2 + [\gamma k(x, y) - 4\pi\sigma^2](\nabla_n\psi_\varphi)^2 = 0. \tag{3.1}$$

By combining (2.8) and (3.1), we have the cubic equation

$$\gamma^2(\nabla_n\psi_\varphi)^6 - (4\pi\sigma^2)^2(\nabla_n\psi_\varphi)^4 - 2\rho\omega^2(4\pi\sigma^2)(\nabla\psi_\varphi)^2 - (\rho\omega^2)^2 = 0,$$

for $(\nabla_n\psi_\varphi)^2$. The exact solution to this equation is difficult. However, by using the short-wavelength approximation, the solution can be expanded in powers of the small parameter $4\pi\sigma^2/\gamma k_0$, where $k_0$ is the characteristic wave number of the problem, and $(\nabla_n\psi_\varphi)^2$ can be separated into a convenient form (2.10) and (2.11), where $n = (1 - 4\pi\sigma^2/\gamma k_0)^{-2/3}$. (In the derivation it is assumed that $\sigma \to 0$ at infinity.) Equation (2.12) is used for the ray trajectory as before.

In particular, this approach makes it possible to describe the focusing properties of nonuniform electric fields in surface hydrodynamics [4].

The equations obtained above allow the geometry of the rays to be determined as they pass through the nonuniformity of the surface stress and the electric field. It is necessary to augment them by the equations for the field amplitude. The method for obtaining the corresponding equations for the electromagnetic waves is shown in [7]. We will use an analogous procedure. We write (2.5) in the form

$$\begin{gathered}-i\omega\rho\varphi_0 \exp(i\psi_\varphi) + (\gamma(\nabla_n\psi_\xi)^2\xi_0 - 4\pi\sigma^2k\xi_0) \times \\ \times \exp(i\psi_\xi) + \{-i\gamma(2\nabla_n\xi_0\nabla_n\psi_\xi + \xi_0\Delta_n\psi_\xi))\} \times \\ \times \exp(i\psi_\xi) - \gamma\Delta_n\xi_0 \exp(i\psi_\xi) = 0,\end{gathered} \tag{3.2}$$

where we group the terms which correspond to the diminution of the power of $\psi$ from two to zero (recall that $\psi$ is a large parameter). Equating the first two terms to zero defines the index of refraction of the surface waves with a consideration of the field. The last term is negligibly small in the limit of geometrical acoustics. The equation for computing the wave amplitude is obtained by setting to zero the second bracket:

$$-i\gamma\left(2\nabla_n\psi\nabla_n\xi_0 + \xi_0\nabla_n^2\psi\right) = 0. \tag{3.3}$$

As in [7], Eq. (3.3) can be used to find the defining conservation law. Actually, we multiply the equation by $\bar{\xi}_0$, combine terms, multiply the resulting equation by $\xi_0$, and then combine the two equations to obtain

$$\nabla_n\psi\nabla_n(\xi_0\bar{\xi}_0) + (\xi_0\bar{\xi}_0)\,\mathrm{div}_n\nabla_n\psi = 0, \tag{3.4}$$

which can be written as

$$\mathrm{div}_n\mathbf{j} = 0,\ \mathbf{j} = (\xi_0\bar{\xi}_0)\nabla_n\psi. \tag{3.5}$$

Thus, we have fulfilled the continuity equation for the vector $\mathbf{j}$, which is proportional to the energy in the wave. As in optics, Eq. (3.4) determines the growth in the wave amplitude during focusing. If we introduce the derivative with respect to the ray direction $\partial/\partial\ell \equiv \psi_n\nabla_n\psi$, then (3.5) takes the form

$$2\partial\xi_0/\partial l + \xi_0\Delta_n\psi = 0.$$

Correspondingly, the product $\xi_0\bar{\xi}_0$ satisfies the equation

$$\partial(\xi_0\bar{\xi}_0)/\partial l + (\xi_0\bar{\xi}_0)\Delta_n\psi = 0,$$

whose solution we write in the form

$$(\xi_0\bar{\xi}_0) = \mathrm{const}\exp\left\{-\int\Delta_n\psi dl\right\}$$

(integration is performed along the ray). The last form of the equation is widely used in optics [7].

It was shown above that the energy flux of the wave is conserved. This is related to the neglect of the wave attenuation. A weak attenuation can be introduced in Eq. (3.2) in analogy to hydrodynamics. This fact and the consideration that the dispersion equation for waves in a low-viscosity fluid is described by (3.3) makes it possible to determine the total change in amplitude related to ray focusing and wave attenuation.

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# EXCITATION OF THERMAL-CAPILLARY CONVECTION AT A DEFORMABLE INTERFACE IN SYSTEMS WITH A SURFACE-ACTIVE AGENT


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The thermal-capillary instability of the layer of a liquid with a free surface has been studied, on which a surface-active agent has been applied [1, 2]. The problem of the initiation of thermal-capillary convection in the presence of a surface-active agent has been solved [3, 4] in the two-layer formulation with a consideration of the hydrodynamic and thermal processes on both sides of the separation surface. In all cases, the problem was examined under the assumption of a plane undeformed interface. It is known that interface deformation can have a significant effect on the excitation of thermal-capillary convection [5-7].

Here, the instability of the equilibrium of systems containing surface-active agents is investigated with a consideration of the deformation of the interface. The effect of the surface-active agent is studied on the monotonic instability mode, and also on oscillatory modes of various types. Features are explained for exciting a special type of oscillatory instability, which is closely related to the presence of a surface-active agent when the interface is deformed.

1. Let the space between two horizontal solid plates at $y = a_1$ and $y = -a_2$, over which a temperature difference $\theta$ is maintained, be filled with two layers of immiscible viscous fluids. The equation of the interface is $y = 0$ in the state of mechanical equilibrium. The densities of the media are $\rho_m$, the coefficients of dynamic and kinematic viscosity are $\eta_m$ and $\nu_m$, the thermal conductivities are $\varkappa_m$, and the heat transfer coefficients are $\chi_m$ ($m = 1$ for the upper layer and $m = 2$ for the lower one). We assume that a surface-active agent is concentrated with a surface (mass) concentration $\Gamma$ at the interface. The concentration of


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